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Topological quantum gate entanglers for a multi-qubit state

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Abstract

We establish a relation between topological and quantum entanglement for a multi-qubit state by considering the unitary representations of the Artin braid group. We construct topological operators that can entangle a multi-qubit state. In particular we construct operators that create quantum entanglement for multi-qubit states based on the Segre ideal of complex multi-projective space. We also discuss in detail and construct these operators for two-qubit and three-qubit states.

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1. Introduction

Multipartite-entangled states are the building blocks of a universal quantum computer. For example, a one-way quantum computer for universal quantum computation is based on entangled cluster states. Recently, Kauffman and Lomonaco have shown that topological entanglement and quantum entanglement are closely related [1]. The authors proposed that it is more fundamental to view topological entanglement as entanglement operators. They also introduced a topological operator called the braiding operator that can entangle quantum states. This operator is a solution of the Yang–Baxter equation, which is also the fundamental topological relation in the Artin braid group and produces a representation of the braid group. That is, if a braiding operator $\mathcal{R}$ satisfies the Yang–Baxter equation and is unitary, then it can be considered as a quantum gate which makes it very suitable for application in the field of quantum computing. We are interested in studying the topological entanglement in the realm of quantum information because quantum computation is more error-prone than classical computation. However, a topological quantum computing scheme makes it possible to construct fault-tolerant quantum computer that is robust against these errors. Thus, quantum information can be protected and processed fault tolerantly. A topological fault-tolerant quantum computation can be constructed by using anyonic systems which are topological modular functors. We have also recently established a relation between multipartite states and the Segre variety and the Segre ideal [2, 3]. For example, we have shown that the Segre ideal represents completely separable states of multipartite states. In this paper, we will construct braiding operators for multi-qubit states based on the construction of the Segre ideal. In particular, in section 2 we will give a short introduction to complex projective variety and complex multi-projective Segre variety and ideal. In section 3 we will review the basic construction of topological entanglement operators. We also discuss the braiding operator for a two-qubit state. Finally, in section 4 we will construct such topological unitary operators for multi-qubit states. We will in detail discuss the construction of these operators for a three-qubit state. Now, denote a general, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}^p_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, with the pure state $|\Psi\rangle = \sum_{k_1,k_2,\ldots,k_m=0}^{N_1-1,N_2-1,\ldots,N_m-1} \alpha_{k_1k_2\ldots k_m}|k_1k_2\cdots k_m\rangle$ and corresponding Hilbert space $\mathcal{H}_\mathcal{Q} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim\left(\mathcal{H}_{\mathcal{Q}_j}\right)$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}^p_2(2, 2)$. Next, let $\rho_\mathcal{Q}$ denote a density operator acting on $\mathcal{H}_\mathcal{Q}$. The density operator $\rho_\mathcal{Q}$ is said to be fully separable, which we will denote by $\rho^{\mathrm{sep}}_\mathcal{Q}$, with respect to the Hilbert space decomposition, if it can be written as $\rho^{\mathrm{sep}}_\mathcal{Q} = \sum_{k=1}^{\mathrm{N}} p_k \bigotimes_{j=1}^m \rho^k_{\mathcal{Q}_j}, \sum_{k=1}^{\mathrm{N}} p_k = 1$ for some positive integer N, where p_k are the positive real numbers and $\rho^k_{\mathcal{Q}_j}$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho^p_\mathcal{Q}$ represents a pure state, then the quantum system is fully separable if $\rho^p_\mathcal{Q}$ can be written as $\rho^{\mathrm{sep}}_\mathcal{Q} = \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is the density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be an entangled state.

2. Complex projective variety and Segre ideal for multi-qubit state

In this section, we will define complex projective space, ideal and variety. Moreover, we will review the construction of the Segre ideal for the multi-qubit state. Here are some general references on complex projective geometry [4, 5]. A complex projective space $\mathbb{P}^{N-1}_\mathbb{C}$ is defined to be the set of lines through the origin in $\mathbb{C}^N$, that is, $\mathbb{P}^{N-1}_\mathbb{C} = \frac{\mathbb{C}^N-0}{(x_0,\ldots,x_{N-1})\sim(y_0,\ldots,y_{N-1})}$, $\lambda \in \mathbb{C} - 0$, $y_i = \lambda x_i\, \forall\, 0 \leqslant i \leqslant N-1$. Let $\mathbb{C}[z] = \mathbb{C}[z_0, z_1, \ldots, z_{N-1}]$ denote the polynomial algebra in N variables with complex coefficients. Then, given a set of homogeneous polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in \mathbb{C}[z]$, we define a complex projective variety as

$$\mathcal{V}(g_1, \ldots, g_q) = \left\{O \in \mathbb{P}^{N-1}_\mathbb{C}\colon g_i(O) = 0\, \forall\, 1 \leqslant i \leqslant q\right\}, \tag{2.1}$$

where $O = [a_0, a_1, \ldots, a_{N-1}]$ denotes the equivalent class of point $\{a_0, a_1, \ldots, a_{N-1}\} \in \mathbb{C}^N$. Let $\mathcal{V}$ be a complex projective variety. Then an ideal of the polynomial algebra $\mathbb{C}[z_0, z_1, \ldots, z_{N-1}]$ is defined by

$$\mathcal{I}(\mathcal{V}) = \{g \in \mathbb{C}[z_0, z_1, \ldots, z_{N-1}]\colon g(z) = 0\, \forall\, z \in \mathcal{V}\}. \tag{2.2}$$

Note also that $\mathcal{V}(\mathcal{I}(\mathcal{V})) = \mathcal{V}$. We can map the product of spaces $\mathbb{P}^{N_1-1}_\mathbb{C} \times \mathbb{P}^{N_2-1}_\mathbb{C} \times \cdots \times \mathbb{P}^{N_m-1}_\mathbb{C}$ into a projective space by its Segre embedding as follows. Let $\left(\alpha^j_0, \alpha^j_1, \ldots, \alpha^j_{N_j-1}\right)$ be points defined on the complex projective space $\mathbb{P}^{N_j-1}_\mathbb{C}$. Then the Segre map

$$\mathbb{P}^{N_1-1}_\mathbb{C} \times \mathbb{P}^{N_2-1}_\mathbb{C} \times \cdots \times \mathbb{P}^{N_m-1}_{\mathbb{C}\mathcal{S}_{N_1,\ldots,N_m}} \mathbb{P}^{N_1N_2\cdots N_m-1}_\mathbb{C} \tag{2.3}$$

is defined by $\left(\left(\alpha^1_0, \alpha^1_1, \ldots, \alpha^1_{N_1-1}\right), \ldots, \left(\alpha^m_0, \alpha^m_1, \ldots, \alpha^m_{N_m-1}\right)\right) \longmapsto \left(\alpha^1_{i_1}\alpha^2_{i_2}\cdots\alpha^m_{i_m}\right)$. Next, let $\alpha_{i_1i_2\cdots i_m}$, $0 \leqslant i_j \leqslant N_j - 1$ be a homogeneous coordinate function on $\mathbb{P}^{N_1N_2\cdots N_m-1}_\mathbb{C}$. For a multi-qubit quantum system the Segre ideal is defined by

$$\mathcal{I}^m_{\text{Segre}} = \sum_{j=1}^{m} \mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m}, \tag{2.4}$$

where $\mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m}$ is the ideal defining when a subsystem $\mathcal{Q}_j$ is separated from the quantum system $\mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$ is generated by

$$\mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m} = \left\langle \text{Minors}_{2\times 2}\mathcal{X}^j_{2\times 2^{m-1}} \right\rangle, \tag{2.5}$$

where $\mathcal{X}^j_{2\times 2^{m-1}}$ is the following $2 \times 2^{m-1}$ matrix:

$$\begin{pmatrix} \alpha_{00\ldots 00_j 0\ldots 0} & \alpha_{00\ldots 00_j 0\ldots 1} & \cdots & \alpha_{11\ldots 10_j 1\ldots 1} \\ \alpha_{00\ldots 01_j 0\ldots 0} & \alpha_{00\ldots 01_j 0\ldots 1} & \cdots & \alpha_{11\ldots 11_j 1\ldots 1} \end{pmatrix}, \tag{2.6}$$

where $j = 1, 2, \ldots, m$ and $\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m$ means we delete $\mathcal{Q}_j$ from the right-hand side and add it to the left-hand side of $\models$.

3. Topological entanglement operators

In this section, we will give a short introduction to the Artin braid group and Yang–Baxter equation. We will study the relation between topological and quantum entanglement by investigating the unitary representation of the Artin braid group. Here are some general references on the quantum group and low-dimensional topology [6, 7]. The Artin braid group B_n on n strands is generated by $\{b_n: 1 \leqslant i \leqslant n-1\}$, and we have the following relations in the group B_n: (i) $b_i b_j = b_j b_i$ for $|i-j| \geqslant n$ and (ii) $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for $1 \leqslant i < n$. Let $\mathcal{V}$ be a complex vector space. Then, for two strand braids there is associated an operator $\mathcal{R}: \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}\otimes\mathcal{V}$. Moreover, let $\mathcal{I}$ be the identity operator on $\mathcal{V}$. Then, the Yang–Baxter equation is defined by

$$(\mathcal{R}\otimes\mathcal{I})(\mathcal{I}\otimes\mathcal{R})(\mathcal{R}\otimes\mathcal{I}) = (\mathcal{I}\otimes\mathcal{R})(\mathcal{R}\otimes\mathcal{I})(\mathcal{I}\otimes\mathcal{R}). \tag{3.1}$$

The Yang–Baxter equation represents the fundamental topological relation in the Artin braid group. The inverse to $\mathcal{R}$ will be associated with the reverse elementary braid on two strands. Next, we define a representation τ of the Artin braid group to the automorphism of $\mathcal{V}^{\otimes m} = \mathcal{V}\otimes\mathcal{V}\otimes\cdots\otimes\mathcal{V}$ by

$$\tau(b_i) = \mathcal{I}\otimes\cdots\otimes\mathcal{I}\otimes\mathcal{R}\otimes\mathcal{I}\otimes\cdots\otimes\mathcal{I}, \tag{3.2}$$

where $\mathcal{R}$ are in position i and $i+1$. This equation describes a representation of the braid group if $\mathcal{R}$ satisfies the Yang–Baxter equation and is also invertible. Moreover, this representation of braid group is unitary if $\mathcal{R}$ is also a unitary operator. Thus $\mathcal{R}$ being unitary indicated that this operator can perform topological entanglement and it can also be considered as a quantum gate. It has been shown in [1] that $\mathcal{R}$ can also perform the quantum entanglement by acting on qubits states. Now, let $\alpha_{00}, \alpha_{01}, \alpha_{10}$ and α_{11} be any scalars on the unit circle in the complex plane. Then, we can construct a unitary $\mathcal{R}$ as follows:

$$\mathcal{R} = \begin{pmatrix} \alpha_{00} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{01} & 0 \\ 0 & \alpha_{10} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{11} \end{pmatrix}, \tag{3.3}$$

which is a solution to the Yang–Baxter equation. To see how it is related to quantum gates, let $\mathcal{P}$ be the swap gate, $\tau = \mathcal{R}\mathcal{P}$, defined by

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tau = \begin{pmatrix} \alpha_{00} & 0 & 0 & 0 \\ 0 & \alpha_{01} & & 0 \\ 0 & 0 & \alpha_{10} & 0 \\ 0 & 0 & 0 & \alpha_{11} \end{pmatrix}. \tag{3.4}$$

In view of braiding and algebra, $\mathcal{R}$ is a solution to the braided version of the Yang–Baxter equation and τ is a solution to the algebraic Yang–Baxter equation and $\mathcal{P}$ represents a virtual or flat crossing. The action of the unitary matrix $\mathcal{R}$ on a quantum state is: (i) $\mathcal{R}|00\rangle = \alpha_{00}|00\rangle$, (ii) $\mathcal{R}|01\rangle = \alpha_{10}|10\rangle$, (iii) $\mathcal{R}|10\rangle = \alpha_{01}|01\rangle$, (iv) $\mathcal{R}|11\rangle = \alpha_{11}|11\rangle$. Note, that the SWAP operator is also a 'NOT' gate that operates on the two bit string b_1b_2 as follows:

$$\mathcal{P}|b_1b_2\rangle = \begin{cases} |b_1b_2\rangle, & \text{if} \quad b_1 = b_2 = 0, \quad \text{or} \quad 1; \\ |\bar{b}_1\bar{b}_2\rangle, & \text{otherwise}, \end{cases} \tag{3.5}$$

where the bar denotes the usual negation, that is $0 \longmapsto 1$ and $1 \longmapsto 0$. A proof that the operator $\mathcal{R}$ can entangle quantum states is give in [1]. Here, we will also give a proof based on the construction of the Segre variety.

Lemma 3.1. *If elements of $\mathcal{R}$ satisfy $\alpha_{00}\alpha_{11} \neq \alpha_{01}\alpha_{10}$, then the state $\mathcal{R}(|+\rangle \otimes |+\rangle)$, with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is entangled.*

From the construction of the Segre ideal the separable set of the two-qubit state satisfies $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$. Thus a two-qubit state

$$\mathcal{R}(|+\rangle \otimes |+\rangle) = \tfrac{1}{2}(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle) \tag{3.6}$$

is entangled if and only if this inequality does not hold. We can also note that a measure of entanglement for the two-qubit state given by concurrence $\mathcal{C}(|\Phi\rangle) = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|$, see [8]. In general, let $M = (\mathcal{M}_{kl})$ denote an $n \times n$ matrix with complex elements and let $\mathcal{R}$ be defined by $\mathcal{R}^{kl}_{rs} = \delta^k_s\delta^l_r\mathcal{M}_{kl}$. Then $\mathcal{R}$ is a unitary solution to the Yang–Baxter equation. In the following section, we will use this construction and proof to create entangled states for three-qubit states.

4. Multi-qubit quantum gate entangler

In the previous section we have shown that how we can create entangled state using topological unitary transformation $\mathcal{R}$. We have also shown a relation between the Segre ideal and such transformation. In this section, we will use this information to construct a multi-qubit-entangled state based on this ideal. But first, we will construct such topological operator for a three-qubit state. For this quantum state the ideal $\mathcal{I}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3}$ is generated by

$$\mathcal{I}_{\mathcal{Q}_1 \models \mathcal{Q}_2\mathcal{Q}_3} = \left\langle \mathrm{Minors}_{2\times 2}\mathcal{X}^1_{2\times 4} \right\rangle = \left\langle \mathrm{Minors}_{2\times 2} \begin{pmatrix} \alpha_{000} & \alpha_{001} & \alpha_{010} & \alpha_{011} \\ \alpha_{100} & \alpha_{101} & \alpha_{110} & \alpha_{111} \end{pmatrix} \right\rangle, \tag{4.1}$$

where we have used the notation $\models$ to indicate that $\mathcal{Q}_1$ is separated from $\mathcal{Q}_2\mathcal{Q}_3$ but we still could have entanglement between $\mathcal{Q}_2$ and $\mathcal{Q}_3$. In the same way, we can define the ideal $\mathcal{I}_{\mathcal{Q}_2 \models \mathcal{Q}_1\mathcal{Q}_3}$ representing if the subsystem $\mathcal{Q}_2$ is unentangled with $\mathcal{Q}_1\mathcal{Q}_3$ and $\mathcal{I}_{\mathcal{Q}_3 \models \mathcal{Q}_1\mathcal{Q}_2}$ representing if the subsystem $\mathcal{Q}_3$ is unentangled with $\mathcal{Q}_2\mathcal{Q}_3$. The ideals are generated by

$\mathcal{I}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3} = \langle \text{Minors}_{2\times 2}\mathcal{X}^2_{2\times 4}\rangle$ and $\mathcal{I}_{\mathcal{Q}_3\models\mathcal{Q}_1\mathcal{Q}_2} = \langle \text{Minors}_{2\times 2}\mathcal{X}^3_{2\times 4}\rangle$. Thus, the Segre ideal for the three-qubit state is given by

$$\begin{aligned} \mathcal{I}^3_{\text{Segre}} &= \mathcal{I}_{\mathcal{Q}_1\models\mathcal{Q}_2\mathcal{Q}_3} + \mathcal{I}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3} + \mathcal{I}_{\mathcal{Q}_3\models\mathcal{Q}_1\mathcal{Q}_2} \\ &= \langle \mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_{12}\rangle, \end{aligned} \tag{4.2}$$

where $\mathrm{T}_1 = \alpha_{000}\alpha_{110} - \alpha_{010}\alpha_{100}$, $\mathrm{T}_2 = \alpha_{001}\alpha_{111} - \alpha_{011}\alpha_{101}$, $\mathrm{T}_3 = \alpha_{000}\alpha_{101} - \alpha_{001}\alpha_{100}$, $\mathrm{T}_4 = \alpha_{010}\alpha_{111} - \alpha_{011}\alpha_{110}$, $\mathrm{T}_5 = \alpha_{000}\alpha_{011} - \alpha_{001}\alpha_{010}$, $\mathrm{T}_6 = \alpha_{100}\alpha_{111} - \alpha_{101}\alpha_{110}$, $\mathrm{T}_7 = \alpha_{000}\alpha_{111} - \alpha_{001}\alpha_{110}$, $\mathrm{T}_8 = \alpha_{000}\alpha_{111} - \alpha_{010}\alpha_{101}$, $\mathrm{T}_9 = \alpha_{000}\alpha_{111} - \alpha_{011}\alpha_{100}$, $\mathrm{T}_{10} = \alpha_{001}\alpha_{110} - \alpha_{010}\alpha_{101}$, $\mathrm{T}_{11} = \alpha_{010}\alpha_{101} - \alpha_{011}\alpha_{100}$, and $\mathrm{T}_{12} = \alpha_{010}\alpha_{101} - \alpha_{011}\alpha_{100}$. In our recent paper [2], we have shown that we can construct a measure of entanglement for three-qubit states based on these Segre varieties. We have also construct a measure of entanglement for general multipartite states based on an extension of the Segre varieties [3]. For example, for the three-qubit state a measure of entanglement is given by

$$\mathcal{C}(|\Psi\rangle) = \left(2\sum_{i=1}^{6}|\mathrm{T}_i|^2 + \sum_{i=7}^{12}|\mathrm{T}_i|^2\right)^{\frac{1}{2}}. \tag{4.3}$$

Now, based on comparison with the two-qubit case we will construct a unitary transformation $\mathcal{R}$ that creates three-qubit entangled states. Let

$$\mathcal{R} = \begin{pmatrix} \alpha_{000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{110} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{101} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{100} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{011} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{010} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{001} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{111} \end{pmatrix}, \tag{4.4}$$

then we have the following lemma for three-qubit states.

Lemma 4.1. *If elements of* $\mathcal{R}$ *satisfy* $\mathrm{T}_i \neq 0$, *for* $1 \leqslant i \leqslant 12$, *then the state* $\mathcal{R}(|+\rangle\otimes|+\rangle\otimes|+\rangle)$, *with* $|+\rangle = \frac{1}{2}(|0\rangle + |1\rangle)$ *is entangled.*

The proof of this lemma follows by the construction of $\mathcal{R}$ which is based on the separable elements of the three-qubit states defined by T_i. For example, $\mathcal{R}(|+\rangle \otimes |+\rangle \otimes |+\rangle) = \sum_{k_1,k_2,k_3=0}^{1} \alpha_{k_1k_2k_3}|k_1k_2k_3\rangle$ is entangled if and only if $T_i \neq 0$. Note that we can also write the braiding operator $\mathcal{R}$ for three-qubit as $\mathcal{R}_{2^3\times 2^3} = \mathcal{R}^d_{2^3\times 2^3} + \mathcal{R}^{ad}_{2^3\times 2^3}$, where $\mathcal{R}^a_{2^3\times 2^3} = (\alpha_{000}, 0, \ldots, 0, \alpha_{111})$ is a diagonal matrix and $\mathcal{R}^{ad}_{2^3\times 2^3} = (0, \alpha_{110}, \alpha_{101}, \ldots, \alpha_{001}, 0)$ is an anti-diagonal matrix. Note also that in this case we have replaced the SWAP operator with a gate that operates on the three bit string $b_1b_2b_3$ as follows:

$$\mathcal{P}|b_1b_2b_3\rangle = \begin{cases} |b_1b_2b_3\rangle, & \text{if} \quad b_1 = b_2 = b_3 = 0, \quad \text{or} \quad 1; \\ |\bar{b}_1\bar{b}_2\bar{b}_3\rangle, & \text{otherwise.} \end{cases} \tag{4.5}$$

We will use this notation to construct the matrix $\mathcal{R}$ for the multi-qubit state. For the m-qubit state a topological unitary transformation $\mathcal{R}_{2^m\times 2^m}$ that creates multi-qubit entangled states is defined by $\mathcal{R}_{2^m\times 2^m} = \mathcal{R}^d_{2^m\times 2^m} + \mathcal{R}^{ad}_{2^m\times 2^m}$, where $\mathcal{R}^a_{2^m\times 2^m} = (\alpha_{0\cdots 0}, 0, \ldots, 0, \alpha_{1\cdots 11})$ is a diagonal matrix and $\mathcal{R}^{ad}_{2^m\times 2^m} = (0, \alpha_{11\cdots 0}, \ldots, \alpha_{10\cdots 0}, \alpha_{01\cdots 1}, \ldots, \alpha_{0\cdots 01}, 0)$ is an anti-diagonal matrix. Then we have the following lemma for general multi-qubit states.

Lemma 4.2. *Let $\mathcal{X}^{j}_{2\times 2^{m-1}}$ be a $2\times 2^{m-1}$ matrix defined by*

$$\mathcal{X}^{j}_{2\times 2^{m-1}} = \begin{pmatrix} \alpha_{00\ldots 00_j 0\ldots 0} & \alpha_{00\ldots 00_j 0\ldots 1} & \cdots & \alpha_{11\ldots 10_j 1\ldots 1} \\ \alpha_{00\ldots 01_j 0\ldots 0} & \alpha_{00\ldots 01_j 0\ldots 1} & \cdots & \alpha_{11\ldots 11_j 1\ldots 1} \end{pmatrix}, \tag{4.6}$$

then the state $\mathcal{R}_{2^m\times 2^m}(|+\rangle_1 \otimes |+\rangle_2 \otimes \cdots \otimes |+\rangle_m)$, with $|+\rangle_j = \frac{1}{2}(|0\rangle + |1\rangle)$ is entangled if 2×2 minors of $\mathcal{X}^{j}_{2\times 2^{m-1}}$ do not vanish, for all $j = 1, 2, \ldots, m$.

The proof follows by the construction of $\mathcal{R}_{2^m\times 2^m}$ which is based on completely separable elements of multi-qubit states defined by the Segre ideal $\mathcal{I}^{m}_{\text{Segre}} = \sum_{j=1}^{m} \mathcal{I}_{\mathcal{Q}_j \models \mathcal{Q}_1\mathcal{Q}_2\cdots\widehat{\mathcal{Q}}_j\cdots\mathcal{Q}_m}$. That is the state $\mathcal{R}_{2^m\times 2^m}(|+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle) = 2^{-m}\sum_{k_1,k_2,\ldots,k_m=0}^{1,1,\ldots,1} \alpha_{k_1k_2\ldots k_m}|k_1k_2\cdots k_m\rangle$ is entangled if and only if all 2×2 minors of $\mathcal{X}^{j}_{2\times 2^{m-1}} \neq 0$. Note that this operator is a quantum gate entangler since $\tau_{2^m\times 2^m} = \mathcal{R}_{2^m\times 2^m}\mathcal{P}_{2^m\times 2^m}$ is a $2^m\times 2^m$ phase gate and $\mathcal{P}_{2^m\times 2^m}$ is a $2^m\times 2^m$ gate that operates on the multi-bit string $b_1b_2\cdots b_m$ as follows:

$$\mathcal{P}|b_1b_2\cdots b_m\rangle = \begin{cases} |b_1b_2\cdots b_m\rangle, & \text{if} \quad b_1 = b_2 = \cdots = b_m = 0, \quad \text{or} \quad 1; \\ |\bar{b}_1\bar{b}_2\cdots\bar{b}_m\rangle, & \text{otherwise.} \end{cases} \tag{4.7}$$

The physical significance of this construction needs further consideration. However, we have succeeded to construct the quantum gate entangler for a general multi-qubit state based on a similar construction of a braiding operator that satisfies the condition for separability that is given by the definition of the Segre ideal. This also shows a good relation between topology, algebraic geometry and quantum theory with application in the field of quantum computing.

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